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On generalized derivations in rings and Banach algebras

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Abstract

Let R be a ring, P a prime ideal, F_1, F_2 are generalized derivations of R and H_1, H_2 two multipliers. The purpose of this paper is to classify the involved mappings satisfying the following differential identity:

$$F_1(x) \perp F_2(y) - H_1(x) \perp H_2(y) \in P \quad \text{for all } x, y \in R,$$

where $\perp$ represents either the usual product or the Lie product $[.,.]$, or the Jordan product $\circ$. Furthermore, as an application, the same identities are studied locally on nonvoid open subsets of prime Banach algebras.

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1 Introduction

Throughout the discussions, R will represent a nonzero associative ring with center $Z(R)$. A ring R is 2-torsion free, in case $2x = 0$, $x \in R$ implies $x = 0$. As usual we write $[x, y]$ for $xy - yx$ and we write $x \circ y$ for $xy + yx$ for $x, y \in R$. Recall that a proper ideal P of R is said to be prime if for any $x, y \in R$, $xRy \subset P$ implies that $x \in P$ or $y \in P$. Therefore, R is called a prime ring if and only if (0) is the prime ideal of R. We denote by Q_r, Q_l, Q_s and C right, left, symmetric Martindale ring of quotients and extended centroid of a prime ring R, respectively, we refer the reader to ([1], Chapter 2) for these objects. An additive mapping $T : R \to R$ is called a left (right) multiplier in case $T(xy) = T(x)y$ ($T(xy) = xT(y)$) holds for all $x, y \in R$. If R has the identity element, $T : R \to R$ is a left (right) multiplier if and only if T is of the form $T(x) = ax$ ($T(x) = xb$) for all $x \in R$ and some fixed elements $a, b \in R$. By d we mean a derivation of R. An additive mapping F from R to R is called a generalized derivation if there exists a derivation d from R to R such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$. The concept of generalized derivation covers both concepts, the concept of derivations and the concept of left multipliers. Since the sum of two generalized derivations is a generalized derivation, every map of the form $F(x) = ax + d(x)$, where a is a fixed element of R and d is a derivation, is a generalized derivation; and if R has 1, all generalized derivations have this form. By a Banach algebra we shall mean a normed algebra whose underlying vector space is a Banach space.

In [8], Hvala showed that if R is a non-commutative ring of characteristic not 2 and F_1, F_2 two generalized derivations of R satisfying $[F_1(x), F_2(x)] = 0$ for all $x \in R$, then there exists $\lambda \in C$ such that $F_1(x) = \lambda F_2(x)$ for all $x \in R$. In the case when the maps are derivations, this and more general conditions were considered by Lanski [9]. We demonstrate that when R is non-commutative, this is possible only when F_1 and F_2 are linearly dependent over an extended centroid. Motivated by this research, the authors in [5] replace the commutator by anti-commutator. More precisely, it's proved that if R is a prime ring of characteristic different from two and F_1, F_2 two generalized derivations of R satisfying $F_1(x) \circ F_2(x) = 0$ for all $x \in R$, then either $F_1 = 0$ or $F_2 = 0$.

When it comes to extending various results on Banach algebras. The authors in [6, Theorem 7] showed that if d is a continuous additive derivation of a Banach algebras $\mathcal{A}$ satisfying $[d(x), d(y)] - [x, y] \in Z(\mathcal{A})$ for all x, y in two nonvoid open subsets of $\mathcal{A}$, then $[\mathcal{A}, \mathcal{A}] \subseteq \mathrm{rad}(\mathcal{A})$. In [7, Theorem 3], the authors investigated more sophisticated identities involving a pair of generalized derivations, more precisely, they proved that if F_1, F_2 are continuous generalized derivations of a noncommutative prime Banach algebra $\mathcal{A}$, O_1, O_2 nonvoid open subsets on $\mathcal{A}$ and n a fixed positive integer such that one of the following identities holds: $(F_1(x) \circ y)^n + x \circ F_2(y) = 0$ for all $(x, y) \in O_1 \times O_2$, $[F_1(x), y]^n + F_2([x, y]) = 0$ for all $x, y \in O_1 \times O_2$, then $F_1 = F_2 = 0$.

This paper aims to study the various differential identities of factor rings and depict the specific form of generalized derivations. Notably, we examine the results mentioned above for generalized derivations involving prime ideals. Our results extend known results for derivations. Moreover, as an application, we investigate continuous generalized derivation satisfying similar algebraic identities locally on nonvoid open subsets of a prime Banach algebra $\mathcal{A}$.

2 Identities with two generalized derivations

In the following, we denote a generalized derivation $F : R \to R$ associated with a derivation d of R by (F, d).

Fact 1. *Let R be a ring and P a prime ideal of R. If (F, d) is a generalized derivation of R such that $\overline{F(x)} \in Z(R/P)$ for all $x \in R$, then $F(R) \subset P$ or R/P is commutative.*

Lemma 1. *Let R be a ring and P a prime ideal of R such that R/P is with characteristic different from two. If (F_1, d_1) and (F_2, d_2) are generalized derivations of R satisfying $[F_1(x), F_2(y)] \in P$ for all $x, y \in R$, then one of the following holds:*
(1) $F_1(P) \subset P$ and $\overline{F_1}(x) = \overline{p_1}\,\overline{x} + \overline{x}\,\overline{p_1}$ for all $x \in R$ and $p_1 \in Q_s(R)$;
(2) $F_2(P) \subset P$ and $\overline{F_2}(x) = \overline{p_2}\,\overline{x} + \overline{x}\,\overline{p_2}$ for all $x \in R$ and $p_2 \in Q_s(R)$;
(3) R/P is commutative.

Proof. Suppose that R/P is not commutative. We have that

$$[F_1(x), F_2(y)] \in P \quad \text{for all} \quad x, y \in R. \tag{1}$$

Replacing y by yr in (1), we get

$$F_2(y)[F_1(x), r] + [F_1(x), y]d_2(r) + y[F_1(x), d_2(r)] \in P \quad \text{for all} \quad x, y, r \in R. \tag{2}$$

Replacing y with ty in (2), where $t \in R$, we find that

$$(F_2(t)y + td_2(y))[F_1(x), r] + [F_1(x), t]yd_2(r) + t[F_1(x), y]d_2(r) + ty[F_1(x), d_2(r)] \in P. \quad (3)$$

Left multiplying (2) by t and subtracting it from (3), we arrive at

$$F_2(t)y[F_1(x), r] + td_2(y)[F_1(x), r] + [F_1(x), t]yd_2(r) - tF_2(y)[F_1(x), r] \in P. \quad (4)$$

As a special case of (4), when $r = F_1(x)$, it follows that

$$[F_1(x), t]Rd_2(F_1(x)) \subset P \quad \text{for all} \quad x, t \in R. \quad (5)$$

P being a prime ideal of R, then we have $[F_1(x), t] \in P$ or $d_2(F_1(x)) \in P$ for all $x, t \in R$. In other words, R is the union of its subsets $R_1 = \{x \in R \,/\, [F_1(x), R] \subset P\}$ and $R_2 = \{x \in R \,/\, d_2(F_1(x)) \in P\}$; note that both of R_1 and R_2 are additive subgroups of R. Thus either $R = R_1$ or $R = R_2$.
Assume that $R = R_1$ i.e., $\overline{F_1(x)} \in Z(R/P)$ for all $x \in R$. Hence, applying Fact 1, we arrive at $F_1(R) \subset P$.
Next, if $R = R_2$, that is

$$d_2(F_1(x)) \in P \quad \text{for all} \quad x \in R. \quad (6)$$

Substituting xy by x in (6), we obtain

$$F_1(x)d_2(y) + d_2(x)d_1(y) + xd_2(d_1(y)) \in P \quad \text{for all} \quad x, y \in R. \quad (7)$$

Writing tx for x in (7), where $t \in R$, we get

$$F_1(t)xd_2(y) + td_1(x)d_2(y) + d_2(t)xd_1(y) + td_2(x)d_1(y) + txd_2(d_1(y)) \in P. \quad (8)$$

Left multiplying (7) by t and subtracting it from (8), we arrive at

$$F_1(t)xd_2(y) + td_1(x)d_2(y) + d_2(t)xd_1(y) - tF_1(x)d_2(y) \in P \quad (9)$$

in such a way that

$$d_2(t)xd_1(F_1(y)) \in P \quad \text{for all} \quad x, y, t \in R. \quad (10)$$

The primeness of P assures that either $d_2(R) \subset P$ or $d_1(F_1(y)) \in P$ for all $y \in R$. Notice that, if $d_2(R) \subset P$, then writing yr for y, it follows that $F_2(y)[F_1(x), r] \in P$ for all $x, y, r \in R$, implying $F_2(R) \subset P$ or $[F_1(x), r] \in P$ for all $x, r \in R$, which proves that $F_1(R) \subset P$ by Fact 1.
Now, using the second case and putting xy instead of y, we require

$$F_1(x)d_1(y) + d_1(x)d_1(y) + xd_1^2(y) \in P \quad \text{for all} \quad x, y \in R. \quad (11)$$

Let $\mathcal{F}(x) = F_1(x) + d_1(x)$. Expression (11) can be written in the form

$$\mathcal{F}(x)d_1(y) + xd_1^2(y) \in P \quad \text{for all} \quad x, y \in R. \quad (12)$$

Substituting yr for y in (12), we find that

$$\mathcal{F}(x)yd_1(r) + 2xd_1(y)d_1(r) + xyd_1^2(r) \in P \quad \text{for all } x, y, r \in R. \quad (13)$$

On the other hand, Eq. (12) yields

$$\mathcal{F}(y)d_1(r) + yd_1^2(r) \in P \quad \text{for all} \quad y, r \in R. \quad (14)$$

Combining (13) with (14), we obtain

$$(\mathcal{F}(x)y + 2xd_1(y) - x\mathcal{F}(y))d_1(r) \in P \quad \text{for all} \quad x, y, r \in R. \tag{15}$$

Let $\mathcal{G}(x) = \mathcal{F}(x) - 2d_1(x)$. The relation (15) can be written as

$$(\mathcal{F}(x)y - x\mathcal{G}(y))d_1(r) \in P \quad \text{for all} \quad x, y, r \in R. \tag{16}$$

Replacing r by rt in (16), we get

$$(\mathcal{F}(x)y - x\mathcal{G}(y))Rd_1(t) \subset P \quad \text{for all} \quad x, y, t \in R. \tag{17}$$

Therefore, we have either $d_1(R) \subset P$ or $\mathcal{F}(x)y - x\mathcal{G}(y) \in P$ for all $x, y \in R$. Consider the second case and putting zx instead of x, we get

$$\mathcal{F}(x)y - x\mathcal{G}(y) \in P \quad \text{for all} \quad x, y \in R. \tag{18}$$

$$\mathcal{F}(zx)y - zx\mathcal{G}(y) \in P \quad \text{for all} \quad x, y, z \in R. \tag{19}$$

Left multiplying (18) by z and subtracting it from (19), we obtain

$$\mathcal{F}(zx) - z\mathcal{F}(x) \in P \quad \text{for all} \quad x, z \in R. \tag{20}$$

It follows that $\mathcal{F}$ is a P-right multiplier.
On the other hand, replacing y by yz in (18), we obviously obtain

$$\mathcal{F}(x)yz - x\mathcal{G}(yz) \in P \quad \text{for all} \quad x, y, z \in R. \tag{21}$$

Combining (18) with (21), it follows that

$$\mathcal{G}(y)z - \mathcal{G}(yz) \in P \quad \text{for all} \quad y, z \in R \tag{22}$$

hence $\mathcal{G}$ is a P-left multiplier.
In other words, in the particular case when $F_1(P) \subset P$, we have $\overline{\mathcal{F}}$ is well defined, so $\overline{\mathcal{F}}$ is a right multiplier on R/P. Thus according to Ref. [5], $\overline{\mathcal{F}}$ is of the form $\overline{\mathcal{F}}(x) = \overline{x}\,\overline{p}$ for all $x \in R$ and some fixed element $p \in Q_r(R)$. Since $\overline{\mathcal{G}}$ is a left multiplier on R/P. Then we have also $\overline{\mathcal{G}}(x) = \overline{q}\,\overline{x}$ for all $x \in R$ and some fixed element $q \in Q_l(R)$.
Whence it follows

$$\overline{q}\,\overline{x} = \overline{\mathcal{G}}(x) = \overline{\mathcal{F}}(x) - 2\overline{d_1}(x) = \overline{x}\,\overline{p} - 2\overline{d_1}(x)$$

in such a way that

$$2\overline{d_1}(x) = \overline{x}\,\overline{p} - \overline{q}\,\overline{x} \quad \text{for all} \quad x \in R. \tag{23}$$

Since $d_1(P) \subset P$, then $\overline{d_1}$ is a derivation of R/P, then we claim that $p = q$. Setting $p_1 = \dfrac{p}{2} \in Q_s(R)$, then we obtain from Eq. (23) along with $char(R/P) \neq 2$ that $\overline{d_1}(x) = \overline{x}\,\overline{p_1} - \overline{p_1}\,\overline{x}$ for all $x \in R$. We have therefore

$$\overline{F_1}(x) = \overline{\mathcal{G}}(x) + \overline{d_1}(x) = \overline{p}\,\overline{x} + \overline{x}\,\overline{p_1} - \overline{p_1}\,\overline{x} = \overline{p_1}\,\overline{x} + \overline{x}\,\overline{p_1} \quad \text{for all} \quad x \in R.$$

Similarly, interchanging the role of F_1 and F_2, one can show that $\overline{F_2}(x) = \overline{p_2}\,\overline{x} + \overline{x}\,\overline{p_2}$ hold for all $x \in R$. ∎

The following result can be proved in a similar way.

Lemma 2. *Let R be a ring and P a prime ideal of R such that R/P is with characteristic different from two. If (F_1, d_1) and (F_2, d_2) are generalized derivations of R satisfying $F_1(x) \circ F_2(y) \in P$ for all $x, y \in R$, then one of the following holds:*
(1) $F_1(P) \subset P$ and $\overline{F_1}(x) = \overline{p_1}\,\overline{x} + \overline{x}\,\overline{p_1}$ for all $x \in R$ and $p_1 \in Q_s(R)$;
(2) $F_2(P) \subset P$ and $\overline{F_2}(x) = \overline{p_2}\,\overline{x} + \overline{x}\,\overline{p_2}$ for all $x \in R$ and $p_2 \in Q_s(R)$.
(3) R/P is commutative. Moreover, we have $F_1(R) \subset P$ or $F_2(R) \subset P$.

Theorem 1. *Let R be a ring and P a prime ideal of R such that R/P is with characteristic different from two. If (F_1, d_1), (F_2, d_2) are generalized derivations of R and H_1, H_2 are multipliers of R satisfying $[F_1(x), F_2(y)] - [H_1(x), H_2(y)] \in P$ for all $x, y \in R$, then one of the following holds:*
(1) $H_1(P) \subset P$, $F_1(P) \subset P$, $\overline{F_1}(x) = \overline{p_1}\,\overline{x} + \overline{x}\,\overline{p_1}$ and $\overline{H_1}(x) = \overline{\lambda_1}\,\overline{x}$ for all $x \in R$ and some elements $p_1 \in Q_s(R)$, $\lambda_1 \in C$;
(2) $H_2(P) \subset P$, $F_2(P) \subset P$, $\overline{F_2}(x) = \overline{p_2}\,\overline{x} + \overline{x}\,\overline{p_2}$ and $\overline{H_2}(x) = \overline{\lambda_2}\,\overline{x}$ for all $x \in R$ and some elements $p_2 \in Q_s(R)$, $\lambda_2 \in C$;
(3) R/P is commutative.

Proof. Suppose that R/P is not commutative. By given assumption, we have

$$[F_1(x), F_2(y)] - [H_1(x), H_2(y)] \in P \quad \text{for all} \quad x, y \in R. \tag{24}$$

Replacing y by yr in (24), where $r \in R$, we get

$$F_2(y)[F_1(x), r] + y[F_1(x), d_2(r)] + [F_1(x), y]d_2(r) - H_2(y)[H_1(x), r] \in P \tag{25}$$

For $r = F_1(x)$ in (25), one can easily see that

$$y[F_1(x), d_2(F_1(x))] + [F_1(x), y]d_2(F_1(x)) - H_2(y)[H_1(x), F_1(x)] \in P \quad \text{for all} \quad x, y \in R. \tag{26}$$

Substituting ty in place of y in (26), we find that

$$[F_1(x), t]Rd_2(F_1(x)) \subset P \quad \text{for all} \quad x, t \in R.$$

By the primeness of P and Fact 1 yields $[F_1(x), t] \in P$ or $d_2(F_1(x)) \in P$ for all $x, t \in R$. The first case forces that $F_1(R) \subset P$. Now if we consider the case $d_2(F_1(R)) \subset P$, then Eq. (26) reduces to

$$H_2(y)[F_1(x), H_1(x)] \in P \quad \text{for all} \quad x, y \in R. \tag{27}$$

Writing ys for y in (27), we obtain

$$H_2(y)R[F_1(x), H_1(x)] \subset P \quad \text{for all} \quad x, y \in R. \tag{28}$$

Again, the primeness of R infers that $H_2(R) \subset P$ or $[F_1(x), H_1(x)] \in P$ for all $x \in R$. The linearization of the second expression leads to

$$[F_1(x), H_1(y)] + [F_1(y), H_1(x)] \in P \quad \text{for all} \quad x, y \in R. \tag{29}$$

Putting $yH_1(x)$ instead y, it follows that

$$y[d_1(H_1(x)), H_1(x)] + [y, H_1(x)]d_1(H_1(x)) \in P \quad \text{for all} \quad x, y \in R. \tag{30}$$

Substituting uy for y in (30), we arrive at

$$[u, H_1(x)]yd_1(H_1(x)) \in P \quad \text{for all} \quad x, y, u \in R.$$

According to the primeness of P along with Fact 1 we get $H_1(R) \subset P$ or $d_1(H_1(x)) \in P$. Now replacing x by xy, we find that

$$P \ni d_1(H_1(xy)) = d_1(H_1(x))y + H_1(x)d_1(y) \quad \text{for all} \quad x, y \in R. \tag{31}$$

This further gives that $H_1(x)d_1(y) \in P$ for all $x, y \in R$. Writing xw for x in the last relation, we obtain $H_1(R) \subset P$ or $d_1(R) \subset P$. Since F_1 and F_2 have a symmetrical role, then we get $F_i(R) \subset P$ or $H_i(R) \subset P$, where $i \in \{1, 2\}$.
Now, if $H_i(P) \subset P$, where $i \in \{1, 2\}$, then according to Ref. [5], we get $\overline{H_1}(x) = \overline{\lambda_1}\,\overline{x}$ and $\overline{H_2}(x) = \overline{\lambda_2}\,\overline{x}$. Otherwise, the main expression becomes

$$[F_1(x), F_2(y)] \in P \quad \text{for all} \quad x, y \in R.$$

Consequently, Lemma 1 completes the proof of our result. ∎

Corollary 1. *Let R be a prime ring with characteristic different from two. If (F, d) is a generalized derivations of R and H_1, H_2 are multipliers of R satisfying $[F_1(x), F_2(y)] = [H_1(x), H_2(y)]$ for all $x, y \in R$, then one of the following holds:*
(1) $F_1(x) = p_1x + xp_1$ and $H_1(x) = \lambda_1 x$ for all $x \in R$ and some elements $p_1 \in Q_s(R)$, $\lambda_1 \in C$;
(2) $F_2(x) = p_2x + xp_2$ and $H_2(x) = \lambda_2 x$ for all $x \in R$ and some elements $p_2 \in Q_s(R)$, $\lambda_2 \in C$;
(3) R is commutative.

The question arises, what can be showed in case when we have a Jordan version? More precisely, we are talking about the relation $F_1(x) \circ F_2(y) - H_1(x) \circ H_2(y) \in P$ for all $x, y \in R$. The above question leads to the following result.

Theorem 2. *Let R be a ring and P a prime ideal of R such that R/P is with characteristic different from two. If (F_1, d_1), (F_2, d_2) are generalized derivations of R and H_1, H_2 are multipliers of R satisfying $F_1(x) \circ F_2(y) - H_1(x) \circ H_2(y) \in P$ for all $x, y \in R$, then one of the following holds:*
(1) $H_1(P) \subset P$, $F_1(P) \subset P$,$\overline{F_1}(x) = \overline{p_1}\,\overline{x} + \overline{x}\,\overline{p_1}$ and $\overline{H_1}(x) = \overline{\lambda_1}\,\overline{x}$ for all $x \in R$ and some elements $p_1 \in Q_s(R)$, $\lambda_1 \in C$;
(2) $H_2(P) \subset P$, $F_2(P) \subset P$,$\overline{F_2}(x) = \overline{p_2}\,\overline{x} + \overline{x}\,\overline{p_2}$ and $\overline{H_2}(x) = \overline{\lambda_2}\,\overline{x}$ for all $x \in R$ and some elements $p_2 \in Q_s(R)$, $\lambda_2 \in C$;
(3) R/P is commutative. Moreover, we have ($F_1(R) \subset P$ or $F_2(R) \subset P$) and ($H_1(R) \subset P$ or $H_2(R) \subset P$).

Corollary 2. *Let R be a non-commutative prime ring with characteristic different from two. If (F_1, d_1), (F_2, d_2) are generalized derivations of R and H_1, H_2 are multipliers of R satisfying $F_1(x) \circ F_2(y) = H_1(x) \circ H_2(y)$ for all $x, y \in R$, then one of the following holds:*
(1) $F_1(x) = p_1x + xp_1$ and $H_1(x) = \lambda_1 x$ for all $x \in R$ and some elements $p_1 \in Q_s(R)$, $\lambda_1 \in C$;
(2) $F_2(x) = p_2x + xp_2$ and $H_2(x) = \lambda_2 x$ for all $x \in R$ and some elements $p_2 \in Q_s(R)$, $\lambda_2 \in C$.

Theorem 3. *Let R be a ring and P a prime ideal of R such that R/P is with characteristic different from two. If (F_1, d_1), (F_2, d_2) are generalized derivations of R and H_1, H_2 are multipliers of R satisfying $F_1(x)F_2(x) - H_1(x)H_2(x) \in P$ for all $x \in R$, then one of the following holds:*
(1) $H_1(P) \subset P$, $F_1(P) \subset P$,$\overline{F_1}(x) = \overline{p_1}\,\overline{x} + \overline{x}\,\overline{p_1}$ and $\overline{H_1}(x) = \overline{\lambda_1}\,\overline{x}$ for all $x \in R$ and some

elements $p_1 \in Q_s(R)$, $\lambda_1 \in C$;
(2) $H_2(P) \subset P$, $F_2(P) \subset P, \overline{F_2}(x) = \overline{p_2}\,\overline{x} + \overline{x}\,\overline{p_2}$ *and* $\overline{H_2}(x) = \overline{\lambda_2}\,\overline{x}$ *for all* $x \in R$ *and some elements* $p_2 \in Q_s(R)$, $\lambda_2 \in C$;
(3) R/P *is commutative. Moreover, we have* ($F_1(R) \subset P$ *or* $F_2(R) \subset P$) *and* ($H_1(R) \subset P$ *or* $H_2(R) \subset P$).

Proof. Analogously, suppose that R/P is not commutative. We have

$$F_1(x)F_2(x) - H_1(x)H_2(x) \in P \quad \text{for all } x \in R. \tag{32}$$

Linearization of (32) gives

$$F_1(x)F_2(y) + F_1(y)F_2(x) - H_1(x)H_2(y) - H_1(y)H_2(x) \in P \quad \text{for all } x, y \in R. \tag{33}$$

Substituting yr for y in (33), where $r \in R$, we get

$$F_1(x)F_2(y)r + F_1(y)rF_2(x) + F_1(x)yd_2(r) + yd_1(r)F_2(x) - H_1(x)H_2(y)r - H_1(y)rH_2(x) \in P. \tag{34}$$

Combining (33) with (34), we arrive at

$$F_1(y)[r, F_2(x)] + F_1(x)yd_2(r) + yd_1(r)F_2(x) - H_1(y)[r, H_2(x)] \in P \quad \text{for all } x, y, r \in R \tag{35}$$

Taking $r = F_2(x)$, one can easily see that

$$F_1(x)yd_2(F_2(x)) + yd_1(F_2(x))F_2(x) + H_1(y)[H_2(x), F_2(x)] \in P \quad \text{for all } x, y \in R. \tag{36}$$

Replacing ty for y in (36), it follows that

$$[F_1(x), t]Rd_2(F_2(x)) \subset P \quad \text{for all } x, t \in R.$$

According to the primeness of P along with Brauer's trick, we have either $[F_1(x), t] \in P$ for all $x, t \in R$ or $d_2(F_2(x)) \in P$ for all $x \in R$.
If $\overline{F_1(x)} \in Z(R/P)$ for all $x \in R$, which makes it possible to conclude that $F_1(R) \subset P$. Henceforth, we shall assume that $F_1(P) \subset P$ from the relation $d_2(F_2(R)) \subset P$ one obtains after some calculations the required result.
Now, if R/P is commutative, then Eq. (35) reduces to

$$F_1(x)d_2(r) + F_2(x)d_1(r) \in P \quad \text{for all } x, r \in R. \tag{37}$$

The substitution xs for x in (37) gives

$$d_1(s)d_2(r) + d_2(s)d_1(r) \in P \quad \text{for all } r, s \in R. \tag{38}$$

Because of $char(R/P) \neq 2$, then one obtains from Eq. (38) that

$$d_1(r)d_2(r) \in P \quad \text{for all } r \in R. \tag{39}$$

Therefore, we deduce that $d_1(R) \subset P$ or $d_2(R) \subset P$, whence, using the same arguments as in the proof of the above relation, it follows that ($F_1(R) \subset P$ or $F_2(R) \subset P$) and ($H_1(R) \subset P$ or $H_2(R) \subset P$). The same procedure as above gives the required result. ∎

Corollary 3. *Let* R *be a non-commutative prime ring with characteristic different from two. If* (F_1, d_1), (F_2, d_2) *are generalized derivations of* R *and* H_1, H_2 *are multipliers of* R *satisfying* $F_1(x)F_2(y) = H_1(x)H_2(y)$ *for all* $x, y \in R$, *then one of the following holds:*
(1) $F_1(x) = p_1x + xp_1$ *and* $H_1(x) = \lambda_1 x$ *for all* $x \in R$ *and some elements* $p_1 \in Q_s(R)$, $\lambda_1 \in C$;
(2) $F_2(x) = p_2x + xp_2$ *and* $H_2(x) = \lambda_2 x$ *for all* $x \in R$ *and some elements* $p_2 \in Q_s(R)$, $\lambda_2 \in C$.

3 Application on prime Banach Algebras

Throughout this section, $\mathcal{A}$ denotes a real or complex Banach algebra. To prove our main results we need the following Lemma.

Lemma 3. [3] *Let $\mathcal{A}$ be a Banach algebra, if $P(t) = \sum_{k=0}^{n} b_k t^k$ is a polynomial in the real variable t with coefficients in $\mathcal{A}$, and if for an infinite set of real values of t, $P(t) \in M$, where M is a closed linear subspace of $\mathcal{A}$, then every b_k lies in M.*

Theorem 4. *Let $\mathcal{A}$ be a prime Banach algebra, O_1, O_2 nonvoid open subsets on $\mathcal{A}$, $Q_{\mathcal{A}}$ its right Martindale quotient ring, $C_{\mathcal{A}}$ its extended centroid, F_1 and F_2 are two continuous generalized derivations of $\mathcal{A}$ associated with derivations d and h respectively, H_1, H_2 are multipliers of $\mathcal{A}$ such that*

$$[F_1(x), F_2(y)] - [H_1(x), H_2(y)] = 0 \text{ for all } x, y \in O_1 \times O_2,$$

for two fixed positive integers $m \geq 1$ and $n \geq 1$. Then one of the following holds:

1. *$F_1(x) = p_1 x + x p_1$ and $H_1(x) = \lambda_1 x$ for all $x \in \mathcal{A}$ and some elements $p_1 \in Q_s(\mathcal{A})$, $\lambda_1 \in C_{\mathcal{A}}$;*
2. *$F_2(x) = p_2 x + x p_2$ and $H_2(x) = \lambda_2 x$ for all $x \in \mathcal{A}$ and some elements $p_2 \in Q_s(\mathcal{A})$, $\lambda_2 \in C_{\mathcal{A}}$;*
3. *$\mathcal{A}$ is commutative.*

Proof. We are given that

$$[F_1(x), F_2(y)] - [H_1(x), H_2(y)] = 0 \text{ for all } (x, y) \in O_1 \times O_2. \quad (40)$$

Let $u \in \mathcal{A}$ and $x \in O_1$, then $x + tu \in O_1$ for a sufficiently small real t.
By assumption F_i, H_i are continuous, then $F_i(ru) = rF_i(u)$ and $H_i(ru) = rH_i(u)$ for all $u \in A$, $r \in \mathbb{R}$, $i \in \{1, 2\}$. Taking $x + tu$ instead of x in equation (40), we get

$$Q(t) := [F_1(x)+tF_1(u), F_2(y)]-[H_1(x)+tH_1(u), H_2(y)] = 0 \text{ for all } (x, y) \in O_1 \times O_2. \quad (41)$$

Setting $Q(t) = \sum_{k=0}^{1} q_k(u, x, y)t^k$. Using Lemma 3, we obtain $q_1(u, x, y) = 0$. That is

$$[F_1(u), F_2(y)] - [H_1(u), H_2(y)] = 0 \text{ for all } (u, y) \in \mathcal{A} \times O_2.$$

Similarly, by acting on y instead of x, one can easily get to

$$[F_1(u), F_2(v)] - [H_1(u), H_2(v)] = 0 \text{ for all } u, v \in \mathcal{A}.$$

By Corollary 1, we get the required results. ∎

Using the same arguments above, with slight modification, application of Corollaries 2 and 3 respectively, we get the following Theorems.

Theorem 5. *Let $\mathcal{A}$ be a prime Banach algebra, O_1, O_2 nonvoid open subsets on $\mathcal{A}$, $Q_{\mathcal{A}}$ its right Martindale quotient ring, $C_{\mathcal{A}}$ its extended centroid, F_1 and F_2 are two continuous generalized derivations of $\mathcal{A}$ associated with derivations d and h respectively, H_1, H_2 are multipliers of $\mathcal{A}$ such that*

$$F_1(x) \circ F_2(y) - H_1(x) \circ H_2(y) = 0 \text{ for all } x, y \in O_1 \times O_2,$$

for two fixed positive integers $m \geq 1$ and $n \geq 1$. Then one of the following holds:

1. *$F_1(x) = p_1x + xp_1$ and $H_1(x) = \lambda_1 x$ for all $x \in \mathcal{A}$ and some elements $p_1 \in Q_s(\mathcal{A})$, $\lambda_1 \in C_{\mathcal{A}}$;*
2. *$F_2(x) = p_2x + xp_2$ and $H_2(x) = \lambda_2 x$ for all $x \in \mathcal{A}$ and some elements $p_2 \in Q_s(\mathcal{A})$, $\lambda_2 \in C_{\mathcal{A}}$;*
3. *$\mathcal{A}$ is commutative.*

Theorem 6. *Let $\mathcal{A}$ be a prime Banach algebra, O_1, O_2 nonvoid open subsets on $\mathcal{A}$, $Q_{\mathcal{A}}$ its right Martindale quotient ring, $C_{\mathcal{A}}$ its extended centroid, F_1 and F_2 are two continuous generalized derivations of $\mathcal{A}$ associated with derivations d and h respectively, H_1, H_2 are multipliers of $\mathcal{A}$ such that*

$$F_1(x)F_2(y) - H_1(x)H_2(y) = 0 \text{ for all } x, y \in O_1 \times O_2,$$

for two fixed positive integers $m \geq 1$ and $n \geq 1$. Then one of the following holds:

1. *$F_1(x) = p_1x + xp_1$ and $H_1(x) = \lambda_1 x$ for all $x \in \mathcal{A}$ and some elements $p_1 \in Q_s(\mathcal{A})$, $\lambda_1 \in C_{\mathcal{A}}$;*
2. *$F_2(x) = p_2x + xp_2$ and $H_2(x) = \lambda_2 x$ for all $x \in \mathcal{A}$ and some elements $p_2 \in Q_s(\mathcal{A})$, $\lambda_2 \in C_{\mathcal{A}}$;*
3. *$\mathcal{A}$ is commutative.*

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Declarations

Conflict of Interest The authors declare that they have no conflict of interest.

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